

QUANTUM KNOT MOSAICS AND THE GROWTH CONSTANT

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ABSTRACT. Lomonaco and Kauffman introduced a knot mosaic system to give a precise and workable definition of a quantum knot system, the states of which are called quantum knots. This paper is inspired by an open question about the knot mosaic enumeration suggested by them. A knot n -mosaic is an $n \times n$ array of 11 mosaic tiles representing a knot or a link diagram by adjoining properly that is called suitably connected. The total number of knot n -mosaics is denoted by D_n which is known to grow in a quadratic exponential rate. In this paper, we show the existence of the knot mosaic constant $\delta = \lim_{n \rightarrow \infty} D_n^{\frac{1}{n^2}}$ and prove that

$$4 \leq \delta \leq \frac{5 + \sqrt{13}}{2} (\approx 4.303).$$

1. PRELIMINARIES

The quantum knot system was developed by Lomonaco and Kauffman to explain how to make quantum information versions of mathematical structures in [5, 6]. They build a knot mosaic system to set the foundation for a quantum knot system, based on the planar projections of knots and the Reidemeister moves.

Throughout this paper the term ‘knot’ means either a knot or a link. An example of a knot mosaic is shown in Figure 1 (a). Knot mosaics are constructed by using 11 mosaic tiles, listed in Figure 1 (b).

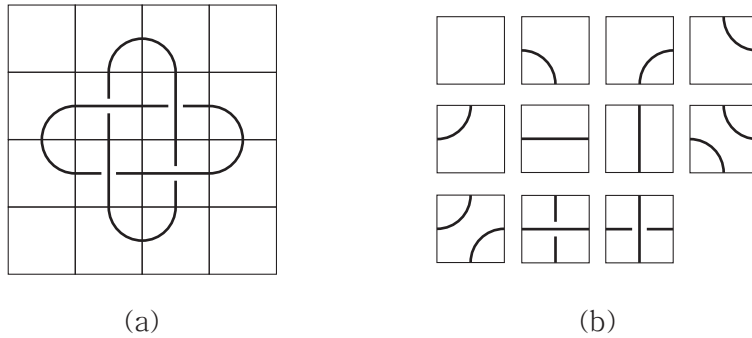


FIGURE 1. An example of a knot mosaic and 11 mosaic tiles

This paper is inspired by an open question (9) about the knot mosaic enumeration proposed in [6]. The enumeration of knot mosaic is not only

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an interesting problem in its own right but is also of considerable importance in the quantum knot theory. Let D_n denote the total number of knot n -mosaics. The author, Hong, Lee and Lee announced several results on D_n in the series of papers [2, 3, 4, 8]. Based upon the results, D_n is known to grow in a quadratic exponential rate. We consider the behavior of the growth rate. The limit, if it exists,

$$\delta = \lim_{n \rightarrow \infty} D_n^{\frac{1}{n^2}}$$

is called the *knot mosaic constant*.

Theorem 1. *The knot mosaic constant δ exists. Furthermore,*

$$4 \leq \delta \leq \frac{5 + \sqrt{13}}{2} (\approx 4.303).$$

As a previous result, lower and upper bounds on D_n for $n \geq 3$ were established as follows in [2]:

$$(*) \quad \frac{2}{275}(9 \cdot 6^{n-2} + 1)^2 \cdot 2^{(n-3)^2} \leq D_n \leq \frac{2}{275}(9 \cdot 6^{n-2} + 1)^2 \cdot (4.4)^{(n-3)^2}$$

These bounds suggested that δ lies between 2 and 4.4.

This paper is organized as follows. In Section 2, we give precise definition of knot mosaics with a slight generalization and previously known results about the enumeration of knot mosaics. In Section 3, the existence of the knot mosaic constant δ is provided by applying an extended version of Fekete's Lemma. In Section 4, we rigorously find lower and upper bounds of δ with heavy reliance on the main theorem of [8].

2. ENUMERATION OF KNOT MOSAICS

We begin by presenting the basic notion of knot mosaics and then introduce previously known results about the enumeration of knot mosaics.

Definition 1 For positive integers m and n , an (m, n) -mosaic is an $m \times n$ array $M = (M_{ij})$ of 11 mosaic tiles depicted in Figure 1 (b).

This definition is a rectangular version of the definition of an n -mosaic that is an $n \times n$ array of mosaic tiles. Obviously the set of all (m, n) -mosaics has 11^{mn} elements.

A *connection point* of a mosaic tile is defined as the midpoint of a mosaic tile edge that is also the endpoint of a curve drawn on the tile. Then the first mosaic tile has zero, the next six tiles with exactly one curve inside have two, and the last four tiles have four connect points. A mosaic is called *suitably connected* if any pair of mosaic tiles lying immediately next to each other in either the same row or the same column have or do not have connection points simultaneously on their common edge.

Definition 2 A *knot (m, n) -mosaic* is a suitably connected (m, n) -mosaic which has no connection point on the boundary edges. $D_{m,n}$ denotes the total number of knot (m, n) -mosaics. Note that $D_n = D_{n,n}$.

A knot (m, n) -mosaic represents a specific knot or link diagram. A knot (n, n) -mosaic is simply specified by a *knot n -mosaic*. Two examples of mosaics are provided in Figure 2.

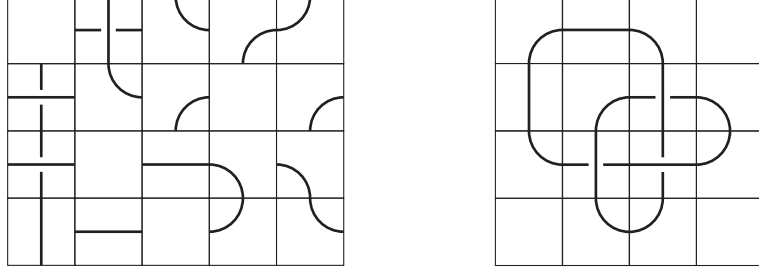


FIGURE 2. Examples of a non-knot $(4, 5)$ -mosaic and the trefoil knot 4-mosaic

The problem of calculating $D_{m,n}$ is one of simplicity of definition but hiding difficulty of solution, due to its non-Markovian processing. $D_{1,n} = 1$ and $D_{2,n} = 2^{n-1}$ for a positive integer n , and $D_{3,3} = 22$. The problem becomes increasingly difficult thereafter. Refer [3] for a table of the precise values of $D_{m,n}$ for $m, n = 4, 5, 6$.

In [8], an algorithm producing the precise value of $D_{m,n}$ was proposed as follows:

Theorem 2 (Oh–Hong–Lee–Lee). *For integers $m, n \geq 3$,*

$$D_{m,n} = 2 \|(X_{m-2} + O_{m-2})^{n-2}\|$$

where X_{m-2} and O_{m-2} are $2^{m-2} \times 2^{m-2}$ matrices recursively defined by

$$X_{k+1} = \begin{bmatrix} X_k & O_k \\ O_k & X_k \end{bmatrix} \text{ and } O_{k+1} = \begin{bmatrix} O_k & X_k \\ X_k & 4O_k \end{bmatrix}$$

for $k = 1, \dots, m-3$, starting with $X_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $O_1 = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$.

Here $\|N\|$ denotes the sum of all entries of a matrix N . Due to Theorem 2, we get Table 1 of the precise values of D_n and approximated values of $D_n^{\frac{1}{n^2}}$. This observation that $D_n^{\frac{1}{n^2}}$ steadily increases is of considerable significance.

3. EXISTENCE OF THE KNOT MOSAIC CONSTANT

In this section, we prove the existence of the knot mosaic constant δ that is the first part of Theorem 1. It is due to the two-variable version of Fekete's Lemma introduced in [7].

Lemma 3 (Two-variable Fekete's Lemma). *A positive valued double sequence $\{a(m, n)\}_{m, n \in \mathbb{N}}$ is superadditive in both indices as $a(m_1 + m_2, n) \geq a(m_1, n) + a(m_2, n)$ and $a(m, n_1 + n_2) \geq a(m, n_1) + a(m, n_2)$. Then*

$$\lim_{n \rightarrow \infty} \frac{a(n, n)}{n^2} = \sup_{n \in \mathbb{N}} \frac{a(n, n)}{n^2}$$

if the supremum exists.

n	D_n	$D_n^{\frac{1}{n^2}}$
1	1	1.000
2	2	1.189
3	22	1.410
4	2594	1.634
5	4183954	1.840
6	101393411126	2.022
7	38572794946976686	2.180
8	234855052870954505606714	2.318
9	23054099362200397056093750003442	2.439

TABLE 1. List of D_n and approximated values of $D_n^{\frac{1}{n^2}}$

Proof. We merely follows the proof of Fekete's Lemma. Let $S = \sup_n \frac{a(n,n)}{n^2}$ and let B be any number less than S . Choose $k \geq 1$ such that $B < \frac{a(k,k)}{k^2}$. For $n > k$, there are integers p_n and q_n such that $n = p_n k + q_n$ and $0 \leq q_n < k$ by the division algorithm. Applying the definition of superadditivity many times in both indices, we obtain

$$\begin{aligned} a(n,n) &\geq p_n a(k, p_n k + q_n) + a(q_n, p_n k + q_n) \\ &\geq p_n^2 a(k,k) + p_n a(k, q_n) + p_n a(q_n, k) + a(q_n, q_n). \end{aligned}$$

So,

$$\frac{a(n,n)}{n^2} \geq \left(\frac{p_n k}{n} \right)^2 \frac{a(k,k)}{k^2}.$$

Since $\frac{p_n k}{n} \rightarrow 1$ as $n \rightarrow \infty$, we have

$$B < \frac{a(k,k)}{k^2} \leq \lim_{n \rightarrow \infty} \frac{a(n,n)}{n^2} \leq S.$$

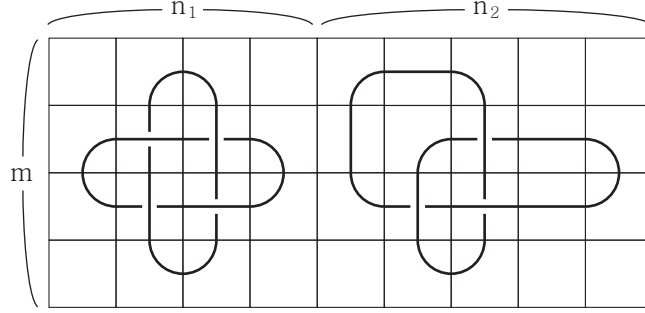
Finally, let B go to S and we obtain

$$\lim_{n \rightarrow \infty} \frac{a(n,n)}{n^2} = S = \sup_{n \in \mathbb{N}} \frac{a(n,n)}{n^2}.$$

□

To show the existence of the limit of $D_n^{\frac{1}{n^2}} = (D_{n,n})^{\frac{1}{n^2}}$, we only need to verify that $D_{m,n}$ satisfies the supermultiplicative property in both indices as $D_{m_1+m_2,n} \geq D_{m_1,n} \cdot D_{m_2,n}$ and $D_{m,n_1+n_2} \geq D_{m,n_1} \cdot D_{m,n_2}$. This asserts that $\log D_{m,n}$ is a superadditive function in both indices. Furthermore the inequality (*) guarantees the existence of $\sup_n \frac{\log D_{n,n}}{n^2} \leq \log 4.4$. Then we can apply Lemma 3.

The supermultiplicative inequalities for $D_{m,n}$ are obvious because we can get a new knot $(m, n_1 + n_2)$ -mosaic by simply adjoining two knot (m, n_1) - and (m, n_2) -mosaics as drawn in Figure 3.

FIGURE 3. $D_{m,n_1+n_2} \geq D_{m,n_1} \cdot D_{m,n_2}$

4. BOUNDS OF THE KNOT MOSAIC CONSTANT

To complete the proof of Theorem 1, we find lower and upper bounds of the knot mosaic constant δ . This procedure heavily relies on Theorem 2.

Define the $2^{n-2} \times 2^{n-2}$ matrices \underline{X}_{n-2} and \underline{Q}_{n-2} by the recurrence relations

$$\underline{X}_{k+1} = \begin{bmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \text{ and } \underline{Q}_{k+1} = \begin{bmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & 4\underline{Q}_k \end{bmatrix}$$

for $k = 1, \dots, n-3$, starting with $\underline{X}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\underline{Q}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$.

Here \mathbb{O} indicates the square zero-matrix with appropriate size. Indeed $(2, 2)$ -quadrant $4O_k$ of O_{k+1} in Theorem 2 has effect on the dominant asymptotic behavior. Then,

$$\underline{D}_n := 2 \|(\underline{X}_{n-2} + \underline{Q}_{n-2})^{n-2}\| \leq D_n.$$

Since the matrix $\underline{X}_{n-2} + \underline{Q}_{n-2}$ has all entries 0, except that its $(2^{n-2}, 2^{n-2})$ -entry is exactly 4^{n-2} ,

$$\underline{\delta} := \lim_{n \rightarrow \infty} \underline{D}_n^{\frac{1}{n-2}} = 4.$$

Furthermore, define the matrices \overline{X}_{n-2} and \overline{O}_{n-2} by

$$\overline{X}_{k+1} = \begin{bmatrix} \overline{X}_k & \overline{O}_k \\ \overline{O}_k & 4\overline{X}_k \end{bmatrix} \text{ and } \overline{O}_{k+1} = \begin{bmatrix} \overline{O}_k & \overline{X}_k \\ \overline{X}_k & 4\overline{O}_k \end{bmatrix}$$

for $k = 1, \dots, n-3$, starting with $\overline{X}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ and $\overline{O}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$. Or, we may start at $k = 0$ with $\overline{X}_0 = [1]$ and $\overline{O}_0 = [1]$. Then,

$$\overline{D}_n := 2 \|(\overline{X}_{n-2} + \overline{O}_{n-2})^{n-2}\| \geq D_n.$$

Considering the matrix $N_k = \overline{X}_k + \overline{O}_k$,

$$\begin{aligned} \|(N_{k+1})^s\| &= \left\| \begin{bmatrix} N_k & N_k \\ N_k & 4N_k \end{bmatrix}^s \right\| = \left\| \begin{bmatrix} a_s(N_k)^s & b_s(N_k)^s \\ c_s(N_k)^s & d_s(N_k)^s \end{bmatrix} \right\| \\ &= (a_s + b_s + c_s + d_s) \|(N_k)^s\| = \left\| \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^s \right\| \cdot \|(N_k)^s\| \end{aligned}$$

where $\begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^s$ for a positive integer s . Then,

$$\begin{aligned} \overline{D}_{n+2} &= 2\|(N_n)^n\| = 2\left\|\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^n\right\| \cdot \|(N_{n-1})^n\| \\ &= 2\left\|\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^n\right\|^n \cdot \|(N_0)^n\| = 2^{n+1} \left\|\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^n\right\|^n. \end{aligned}$$

Owing to the equality $\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} = P \cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot P^{-1}$ where λ_1 and λ_2 are the eigenvalues $\frac{5-\sqrt{13}}{2}$ and $\frac{5+\sqrt{13}}{2}$, respectively, of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ and P is the diagonalizing matrix,

$$\begin{aligned} \overline{\delta} &:= \lim_{n \rightarrow \infty} \overline{D}_n^{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \overline{D}_{n+2}^{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \left\|\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^n\right\|^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \left\|P \cdot \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \cdot P^{-1}\right\|^{\frac{1}{n}} = \lambda_2 = \frac{5+\sqrt{13}}{2}. \end{aligned}$$

As a conclusion, the inequality $\underline{D}_n \leq D_n \leq \overline{D}_n$ guarantees the following bounds of the knot mosaic constant as desired,

$$\underline{\delta} \leq \delta \leq \overline{\delta}.$$

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